Continuity equation

R.T.T.: $\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} + \int_{cs} \rho b \vec{V}\cdot\vec{n}\, dA$

Let $B_{sys} = M_{sys}$ the mass of the system

then $b = 1$

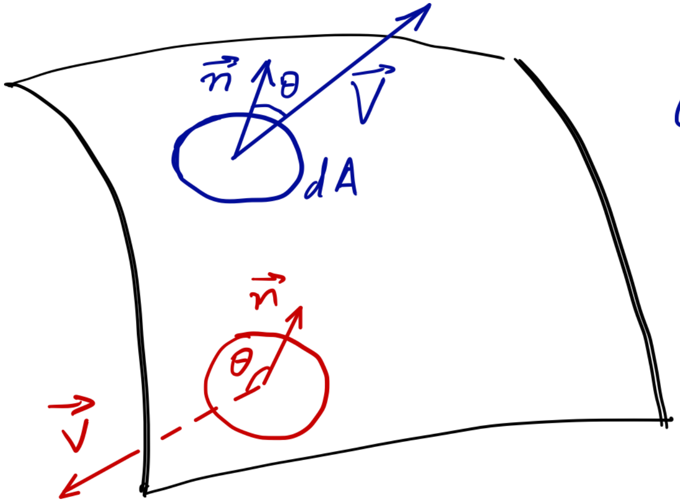
Conservation of mass principle: $\frac{DM_{sys}}{Dt} = 0$

Hence: $\frac{\partial M_{cv}}{\partial t} + \int_{cs} \rho \vec{V}\cdot\vec{n}\, dA = 0$

or

$$\underbrace{\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall}_{\substack{\text{time rate of} \\ \text{change of mass} \\ \text{inside the C.V.}}} + \underbrace{\int_{cs} \rho \vec{V}\cdot\vec{n}\, dA}_{\substack{\text{net rate of flow} \\ \text{of mass through} \\ \text{the control surface}}} = 0$$

Conservation of mass



$\theta < 90^\circ$ $\quad \vec{V}\cdot\vec{n} > 0$ for flow out

$\theta > 90^\circ$ $\quad \vec{V}\cdot\vec{n} < 0$ for flow in

$$\int_{CS} \underbrace{\rho \underbrace{\vec{V}\cdot\vec{n}\, dA}_{\text{flowrate through } dA}}_{\text{mass flowrate through } dA} = \sum \dot{m}_{out} - \sum \dot{m}_{in}$$

Rewrite as

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \Sigma \dot m_{out} - \Sigma \dot m_{in} = 0$$

For steady flows $\frac{\partial}{\partial t}(\ ) = 0$

So, $$\Sigma \dot m_{out} - \Sigma \dot m_{in} = 0$$

or $$\Sigma \dot m_{out} = \Sigma \dot m_{in}$$

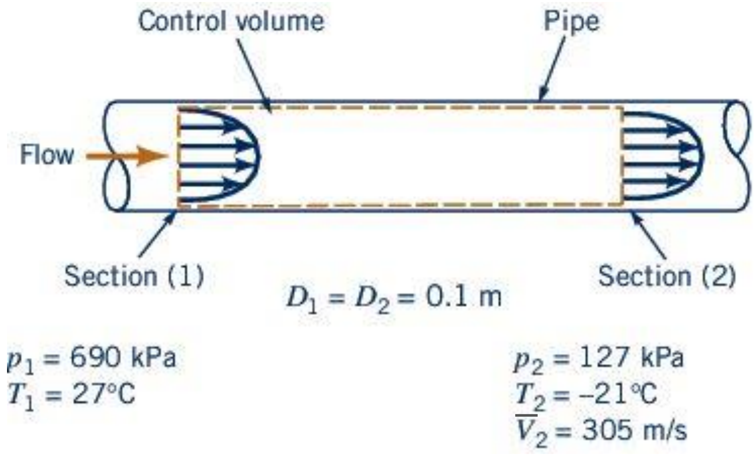
The mass flowrate, $\dot m$, is also given as

$$\dot m = \underset{\substack{\uparrow \\ \text{density}}}{\rho} \cdot \underset{\nwarrow\ \text{flowrate}}{Q} = \rho \cdot \underbrace{V \cdot A}_{\text{only when } V \text{ is } \underline{\text{constant}} \text{ across } A \text{ and } \underline{\text{perpendicular}} \text{ to area } A}$$

If V is not uniform, use the average velocity $\overline{V}$

$$\overline{V} = \frac{\int_A \rho \vec{V} \cdot \vec{n} \, dA}{\rho A} \qquad \dot m = \rho \cdot \overline{V} \cdot A$$

Example 1: Continuity in compressible fluid flow



Continuity: $$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall}}_{\to 0 \text{ (steady flow)}} + \dot{m}_{out} - \dot{m}_{in} = 0$$

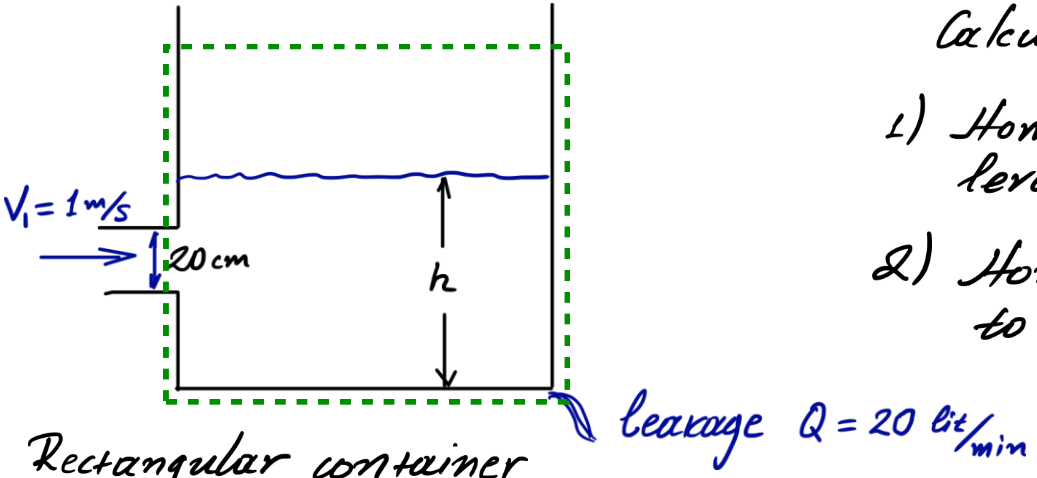
$$\Rightarrow \quad \dot{m}_{out} = \dot{m}_{in} \Rightarrow \rho_1 \bar{V}_1 A_1 = \rho_2 \bar{V}_2 A_2$$

and since $A_1 = A_2$: $$\rho_1 \bar{V}_1 = \rho_2 \bar{V}_2 \Rightarrow \bar{V}_1 = \frac{\rho_2}{\rho_1}\bar{V}_2$$

From ideal gas law: $$\rho = \frac{P}{RT}$$

Hence, $$\bar{V}_1 = \frac{P_2 T_1 \bar{V}_2}{P_1 T_2} = \frac{127\,kPa}{690\,kPa}\cdot\frac{300\,K}{252\,K}\,305\,m/s \Rightarrow \underline{\underline{\bar{V}_1 = 66.8\,m/s}}$$

Example 2: continuity in unsteady flow



Rectangular container
0.6m × 0.6m

Calculate:

1) How fast the water level rises?

2) How long does it take to rise 0.5 m?

$$\text{Continuity:}\quad \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \sum \dot{m}_{out} - \sum \dot{m}_{in} = 0 \qquad \rho = ct$$

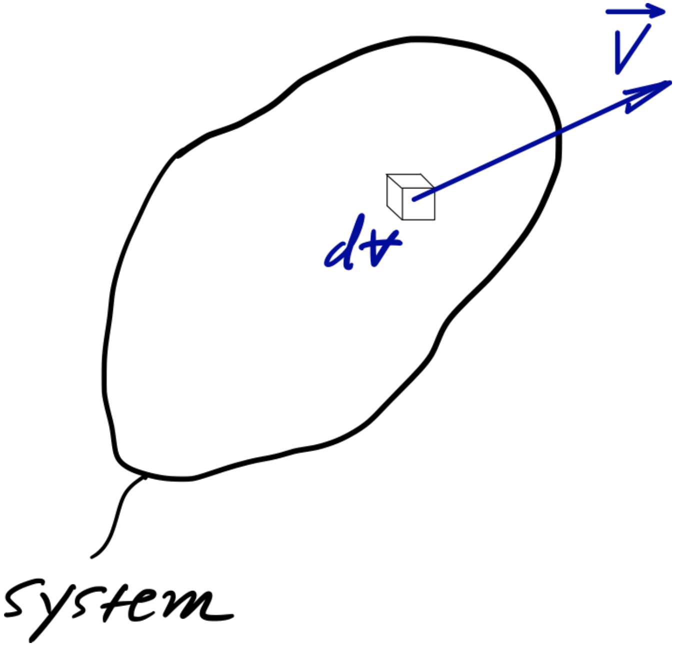
$$\cancel{\rho}\,\frac{\partial}{\partial t}\int_{CV} d\forall + \cancel{\rho}\,Q_{out} - \cancel{\rho}\,V_1 A_1 = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(0.6 \times 0.6 \times h\right) + 20\,lit/min \cdot \frac{10^{-3}\, m^3/lit}{60\ s/min} - 1\, m/s \times \pi\, 0.1^2\, m^2 = 0$$

$$\Rightarrow 0.36\,\frac{dh}{dt} + 3.33 \times 10^{-4} - 0.0314 = 0 \Rightarrow \underline{\underline{\frac{dh}{dt} = 0.0311\ m/s}}$$

$$t = \frac{\Delta h}{dh/dt} = \frac{0.5\,m}{0.0311\ m/s} = \underline{\underline{16.1\ s}}$$

Linear momentum equation



Newton's 2nd law:

$$\frac{D}{Dt}\int_{sys}\underbrace{\vec{V}\cdot\underbrace{\rho\, d\forall}_{dm}}_{\text{momentum in } d\forall} = \sum \vec{F}_{sys}$$

Apply the RTT $\left(B = m\vec{V} \Rightarrow b = \vec{V}\right)$:

$$\frac{\partial}{\partial t}\int_{CV}\vec{V}\rho\, d\forall + \int_{CS}\vec{V}\rho\,\vec{V}\cdot\vec{n}\, dA = \underbrace{\sum \vec{F}_{CV}}_{\substack{\text{Sum of}\\ \text{body and}\\ \text{surface forces}\\ \text{acting on the}\\ \text{control volume}}}$$

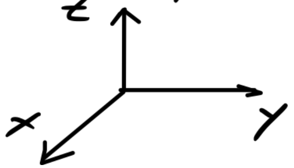
Linear momentum equation

Linear momentum in x, y & z

When the flow is steady $\left(\frac{\partial}{\partial t}(\ )=0\right)$:

$$\int_{CS} \vec{V}\rho\, \underbrace{\vec{V}\cdot\vec{n}}_{\text{scalar}}\, dA = \Sigma \vec{F}_{CV}$$

Vector equation



$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

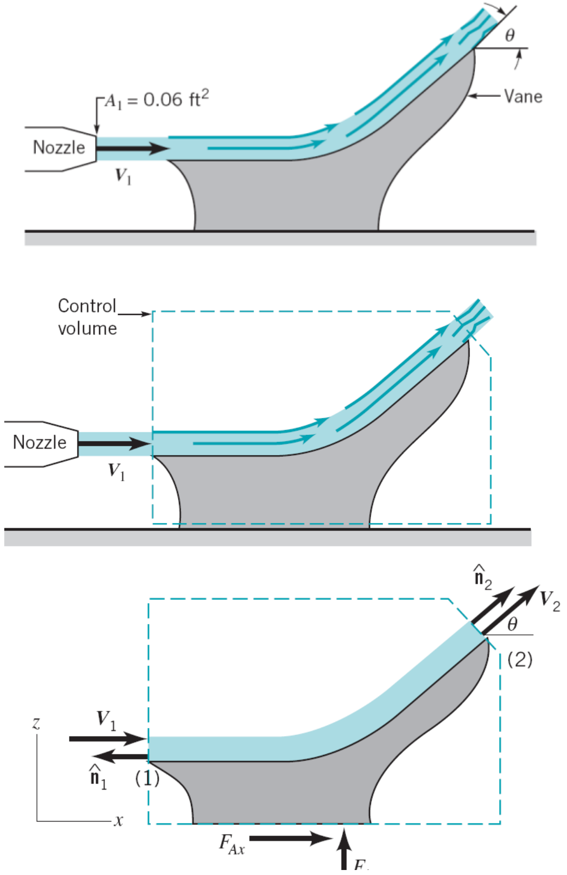
So, the linear momentum equation in x, y & z:

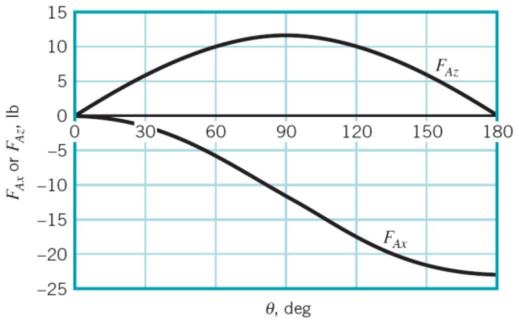
x: $$\int_{CS} u\cdot\rho\, \vec{V}\cdot\vec{n}\, dA = \Sigma F_x$$

y: $$\int_{CS} v\rho\, \vec{V}\cdot\vec{n}\, dA = \Sigma F_y$$

z: $$\int_{CS} w\rho\, \vec{V}\cdot\vec{n}\, dA = \Sigma F_z$$

Example: linear momentum - change in flow direction





Linear momentum in x and z directions:

$$x: \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}}_{0\ (\text{steady})} + \int_{CS} u\rho\, \vec{V}\cdot\vec{n}\, dA = \sum F_x$$

$$z: \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall}}_{0\ (\text{steady})} + \int_{CS} w\rho\, \vec{V}\cdot\vec{n}\, dA = \sum F_z$$

or $V_1\cdot\rho\cdot(-V_1)A_1 + (V_2\cos\theta)\cdot\rho\cdot V_2\cdot A_2 = F_{Ax}$

and $0 + (V_2\sin\theta)\cdot\rho\cdot V_2\cdot A_2 = F_{Az}$

$P=0$ (atm)
No pressure forces

From continuity $A_1V_1 = A_2V_2$.

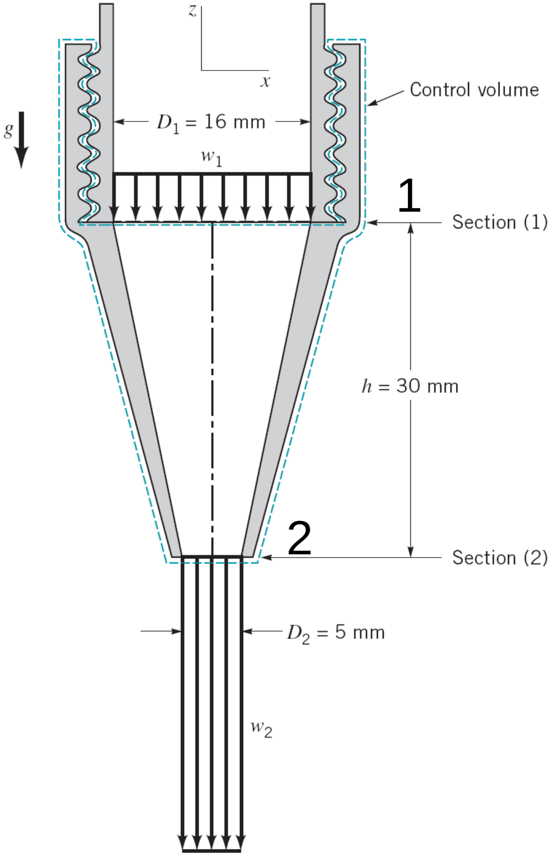
Bernoulli: $\underbrace{\cancel{P_1}}_{0} + \frac{1}{2}\rho V_1^2 + \gamma z_1 = \underbrace{\cancel{P_2}}_{0} + \frac{1}{2}\rho V_2^2 + \gamma z_2$

since $z_2 \simeq z_1 \Rightarrow V_1 = V_2$

Hence: $F_{Ax} = -\rho A_1 V_1^2 (1-\cos\theta)$

and $F_{Az} = \rho A_1 V_1^2 \sin\theta$

Linear momentum: weight, pressure and speed change



Flowrate: 0.6 lit/s

Nozzle mass: 0.1 kg

$P_1 = 464$ kPa

What is the anchoring force to keep the nozzle in place?

From continuity:

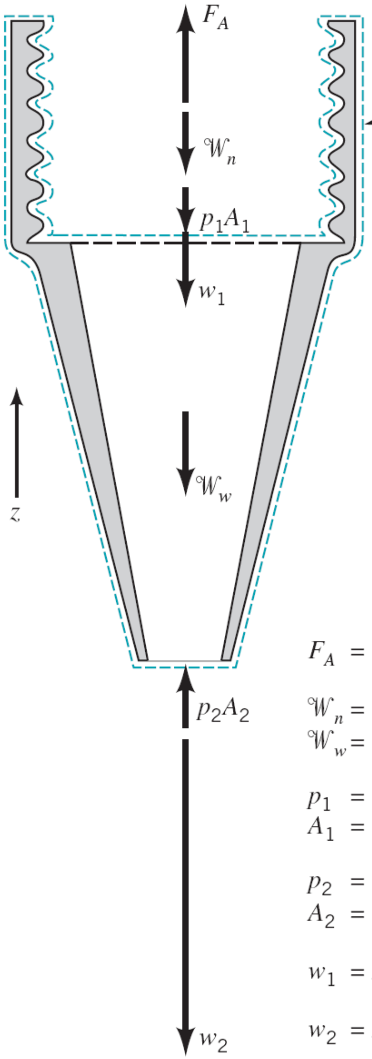
$$Q = w_1 A_1 = w_2 A_2$$

$$w_1 = \frac{Q}{A_1} = \frac{0.6\times10^{-3}\ m^3/s}{\pi\times 0.008^2\ m^2}$$

$$\underline{w_1 = 2.98\ m/s}$$

$$w_2 = \frac{A_1}{A_2} w_1 = \left(\frac{16}{5}\right)^2 2.98\ m/s$$

$$\underline{w_2 = 30.5\ m/s}$$



F_A = anchoring force that holds nozzle in place
$\mathscr{W}_n$ = weight of nozzle
$\mathscr{W}_w$ = weight of water contained in the nozzle
p_1 = gage pressure at section (1)
A_1 = cross section area at section (1)
p_2 = gage pressure at section (2)
A_2 = cross section area at section (2)
w_1 = z direction velocity at control volume entrance
w_2 = z direction velocity at control volume exit

z-momentum:

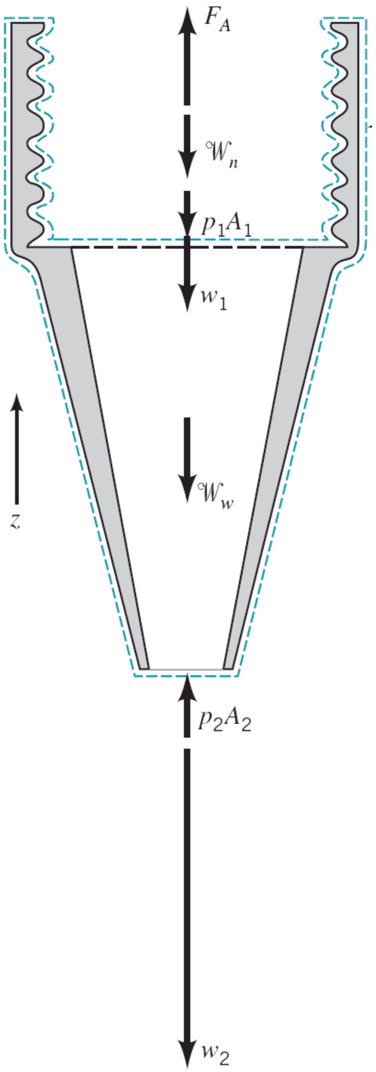
$$\int_{cs} w \rho \vec{V}\cdot\vec{n}\, dA = F_A - W_n - P_1 A_1 - W_w + \cancelto{0}{P_2} A_2$$

$$\left.\begin{array}{l} \Rightarrow (-w_1)\rho(-w_1)A_1 + (-w_2)\rho(w_2)A_2 = F_A - W_n - P_1A_1 - W_w \\ \text{Mass flowrate } \dot{m} = \rho W_1 A_1 = \rho W_2 A_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \dot{m} W_1 - \dot{m} W_2 = F_A - W_n - P_1 A_1 - W_w$$

$$\Rightarrow \underline{\underline{F_A = \dot{m}(w_1 - w_2) + P_1 A_1 + W_n + W_w}}$$

$$\dot{m} = \rho \cdot Q = 999\ kg/m^3 \cdot 0.6\ lit/s \cdot \frac{1\ m^3}{1000\ lit} = \underline{0.599\ kg/s}$$



$$W_n = m_n \cdot g = 0.1\,kg \cdot 9.81\,m/s^2 = \underline{0.981\,N}$$

$$W_w = \rho \cdot V_w \cdot g$$

The volume of water, V_w, is calculated as the volume of the truncated cone:

$$V_w = \frac{1}{12}\pi h\left(D_1^2 + D_2^2 + D_1 D_2\right)$$
$$= \frac{1}{12}\pi\, 0.03\left(0.016^2 + 0.005^2 + 0.016 \times 0.005\right)$$
$$= 2.84 \times 10^{-6}\,m^3$$

Hence, $W_w = 999\,kg/m^3 \times 2.84 \times 10^{-6}\,m^3 \times 9.81\,m/s^2$

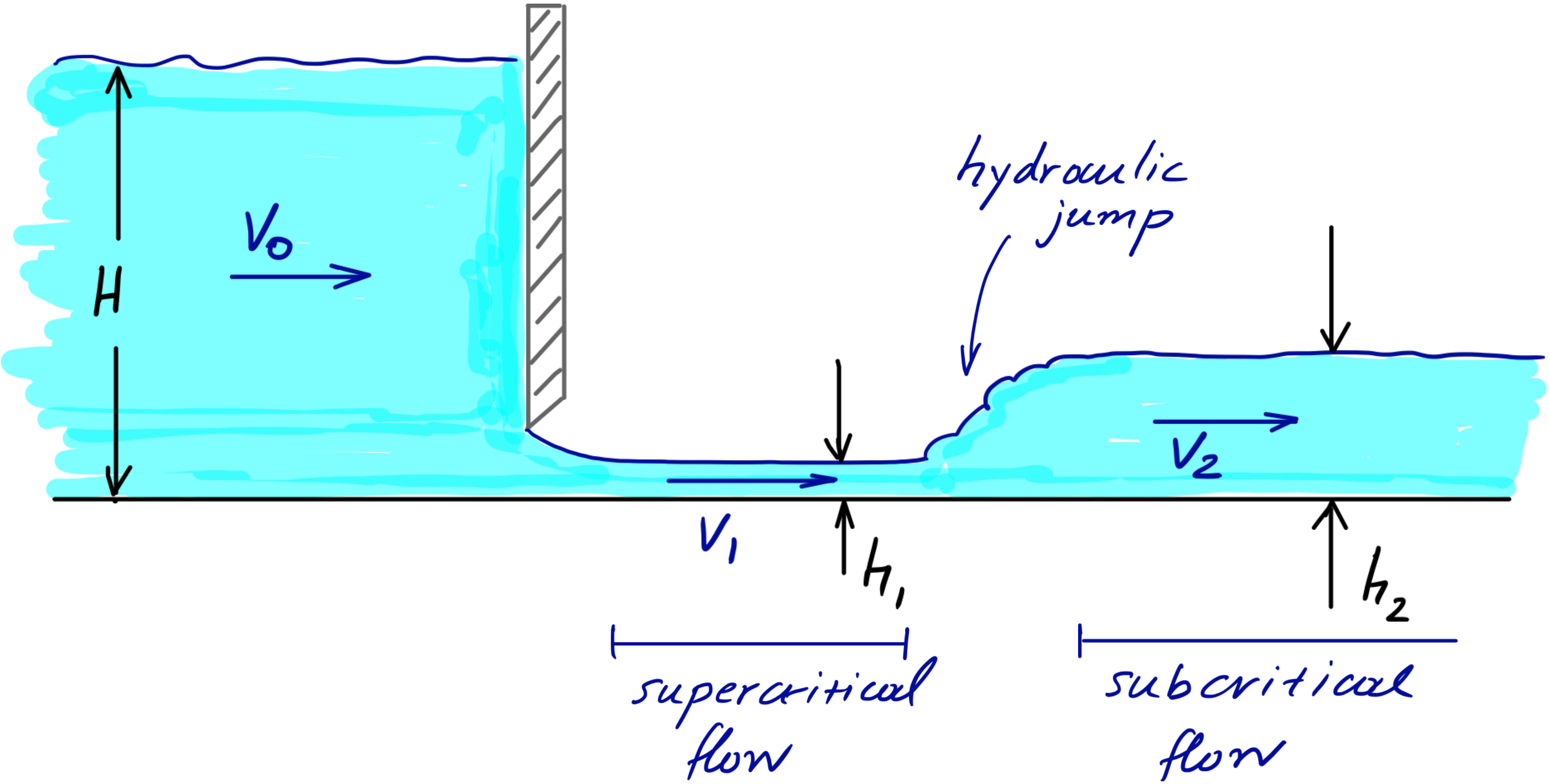
$$= \underline{0.0278\,N}$$

Finally, the anchoring force, F_A, is calculated as:

$$F_A = \dot{m}\left(w_1 - w_2\right) + P_1 A_1 + W_n + W_w$$
$$= 0.599\,kg/s\left(2.98\,m/s - 30.5\,m/s\right) + 464'000\,N/m^2 \times \pi\, 0.008^2\,m^2$$
$$+\ 0.981\,N + 0.0278\,N$$

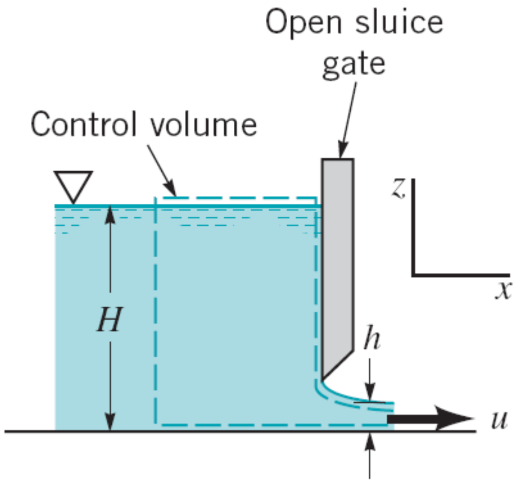
$$\Rightarrow \quad \underline{\underline{F_A = 77.8\,N}}$$

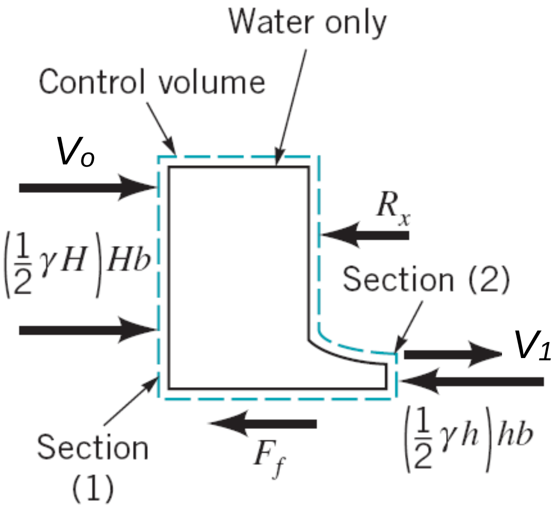
Example: sluice gate



Sluice gate hydraulic jump experiment (video)

Part 1: force on sluice gate





Momentum (steady flow):

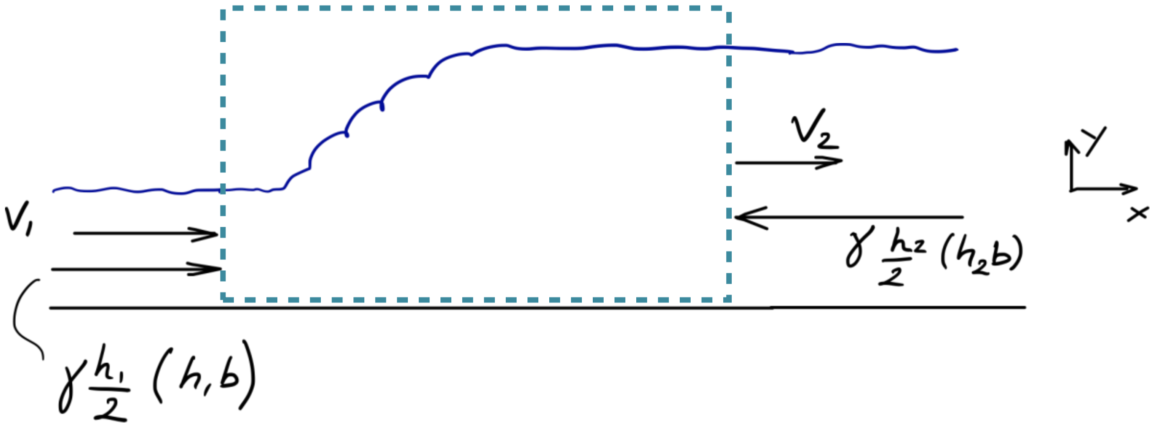
$$\int_{CS} u\rho\,\vec{V}\cdot\vec{n}\,dA = \Sigma F_x$$

$$\Rightarrow V_0\rho(-V_0)A_0 + V_1\rho V_1 A_1 = \gamma\frac{H}{2}(Hb) - \gamma\frac{h_1}{2}(h_1 b) - R_x$$

If $h_1 \ll H \rightarrow V_0 \ll V_1$, then

$$R_x = \underbrace{\frac{1}{2}\gamma H^2 b}_{\text{force on sluice gate when fully closed}} - \rho V_1^2 b h_1$$

Part 2: height ratio across the hydraulic jump



$$\int_{cs} u\rho\, \vec{V}\cdot\vec{n}\, dA = \frac{1}{2}\gamma h_1^2 b - \frac{1}{2}\gamma h_2^2 b$$

$$\Rightarrow \quad V_1 \rho (-V_1) A_1 + V_2 \rho (V_2) A_2 = \frac{1}{2}\gamma (h_1^2 - h_2^2) b$$

$$\Rightarrow \quad -\rho V_1^2 h_1 b + \rho V_2^2 h_2 b = \frac{1}{2}\gamma (h_1^2 - h_2^2) b$$

Divide by $\rho \cdot b$

$$\Rightarrow \quad V_2^2 h_2 - V_1^2 h_1 = \frac{1}{2} g (h_1^2 - h_2^2) \qquad (1)$$

From continuity, $V_1 h_1 b = V_2 h_2 b \Rightarrow V_2 = V_1 \frac{h_1}{h_2}$

$$(1) \Rightarrow \quad V_1^2 \frac{h_1^{\cancel{2}}}{h_2} - V_1^2 \cancel{h_1} = \frac{1}{2} g h_1^{\cancel{2}} \left(1 - \frac{h_2^2}{h_1^2}\right)$$

$$\Rightarrow \quad V_1^2 \left(\frac{h_1}{h_2} - 1\right) = \frac{1}{2} g h_1 \left(1 - \frac{h_2}{h_1}\right)\left(1 + \frac{h_2}{h_1}\right)$$

$$\Rightarrow \quad V_1^2 \frac{h_1}{h_2}\left(1 - \cancel{\frac{h_2}{h_1}}\right) = \frac{1}{2} g h_1 \left(1 - \cancel{\frac{h_2}{h_1}}\right)\left(1 + \frac{h_2}{h_1}\right)$$

$$\Rightarrow \quad \frac{V_1^2}{g h_1} = \frac{1}{2} \frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right) \qquad \text{let } F_r = \frac{V_1}{\sqrt{g h_1}}$$

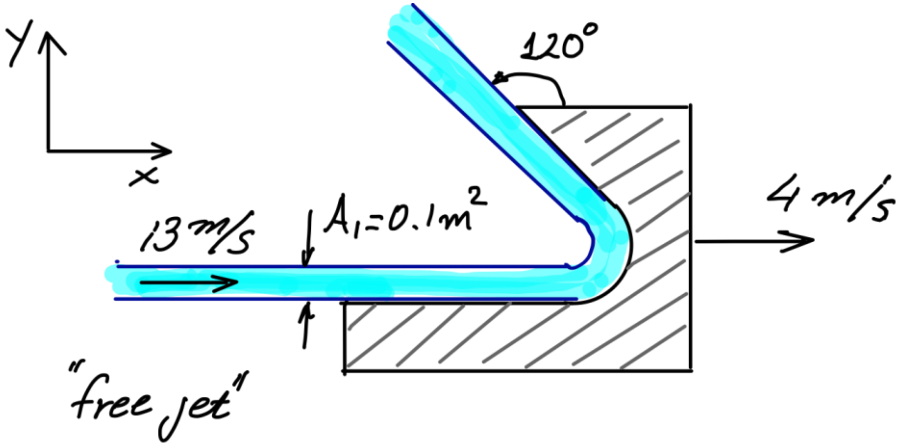
Froude number

$$\Rightarrow \quad \left(\frac{h_2}{h_1}\right)^2 + \frac{h_2}{h_1} - 2 F_r^2 = 0$$

$$\Rightarrow \quad \frac{h_2}{h_1} = \frac{-1 \pm \sqrt{1 + 8 F_r^2}}{2} \qquad (\text{keep + sign})$$

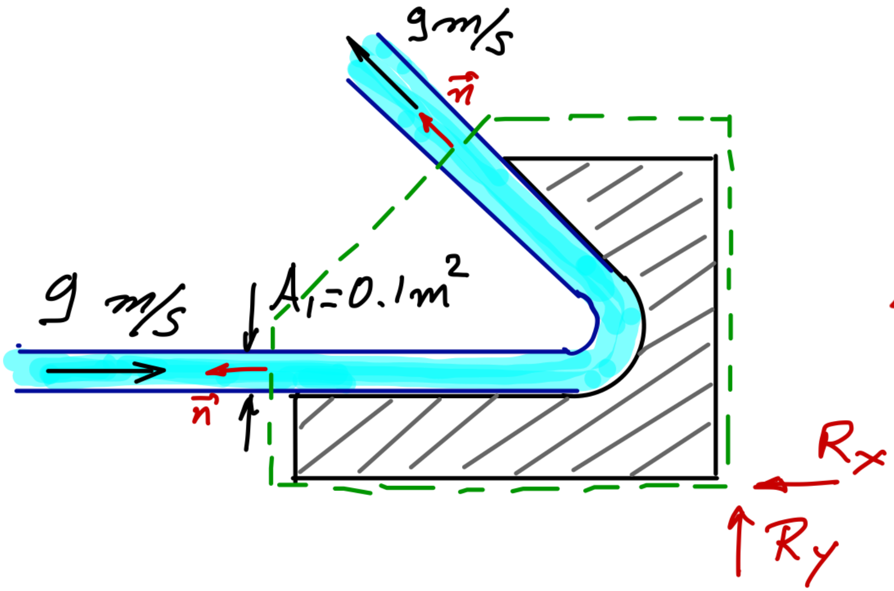
$$\Rightarrow \quad \underline{\underline{\frac{h_2}{h_1} = \frac{1}{2}\left[\sqrt{1 + 8 F_r^2} - 1\right]}}$$

Example: force on a moving vane



What is the force that the water jet exerts <u>on the vane</u>?

For an observer on the moving vane, the situation looks as follows:



"Free jet"

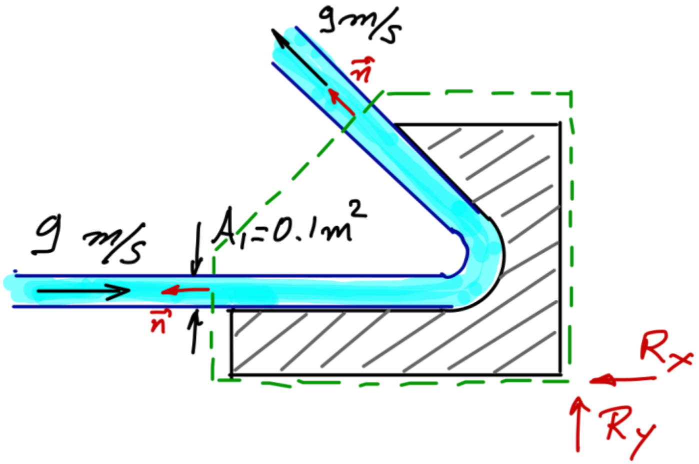
→ P is 0 (atmospheric) everywhere in the jet

$$\underset{\to 0}{\cancel{P_1}} + \frac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = \underset{\to 0}{\cancel{P_2}} + \frac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}$$

$$\Rightarrow \quad V_1 = V_2 = 9\ m/s$$

Continuity: $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ $(\rho_1 = \rho_2)$

$$\Rightarrow \quad A_2 = A_1 = 0.01\ m^2$$



x: $$\int_{cs} u \rho \vec{V}\cdot\vec{n}\, dA = \Sigma F_x$$

$$V_1 \rho (-V_1) A_1 - V_2 \cos 60 \rho (V_2) A_2 = -R_x$$

$$\Rightarrow R_x = \rho V_1^2 A_1 + \cos 60 \rho V_2^2 A_2$$

$$= (1 + \cos 60) \rho V_1^2 A_1$$

$$= 1.5 \times 999 \times 9^2 \times 0.1 \text{ N}$$

$$= \underline{\underline{1210 \text{ N}}}$$ (to the left)

y: $$\int_{cs} v \rho \vec{V}\cdot\vec{n}\, dA = \Sigma F_y$$

$$\Rightarrow V_2 \sin 60 \cdot \rho \cdot V_2 A_2 = R_y$$

$$\Rightarrow R_y = \sin 60 \rho V_2^2 A_2$$

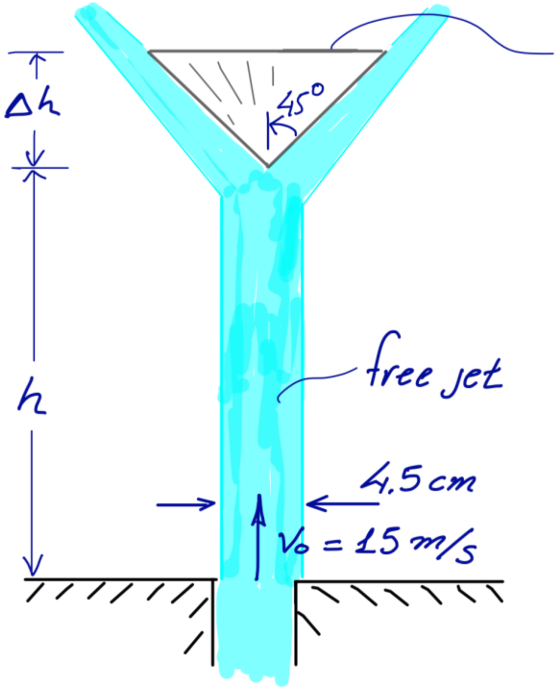
$$= \sin 60 \cdot 999 \cdot 9^2 \cdot 0.01 \text{ N}$$

$$= \underline{\underline{701 \text{ N}}}$$ (up)

The anchoring force to keep the vane in place:

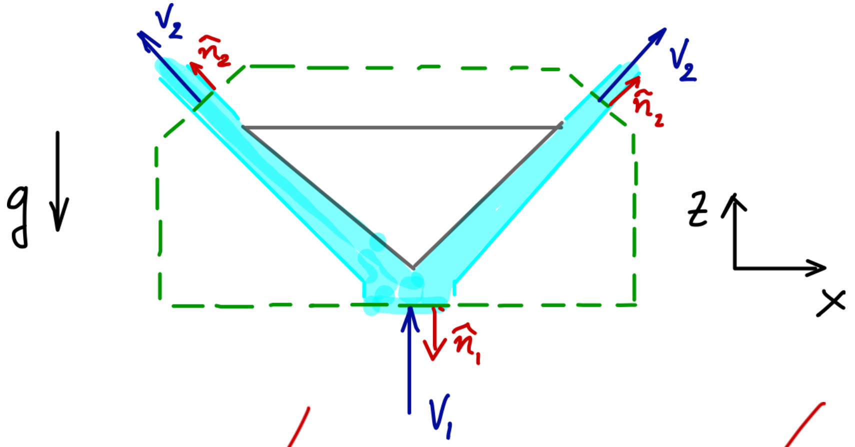
The force the water jet exerts on the vane:

Cone suspended by a free water jet



90° cone
weight, $W_c = 50N$

What is the suspension height, h?
Assume $\Delta h \ll h$



Bernoulli:

$$\underbrace{\cancel{p_1}}_{\to 0 \text{ free jet}} + \frac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = \underbrace{\cancel{p_2}}_{\to 0 \text{ free jet}} + \frac{1}{2}\rho V_2^2 + \cancel{\gamma(\underbrace{z_1+\Delta h}_{\simeq z_1})}$$

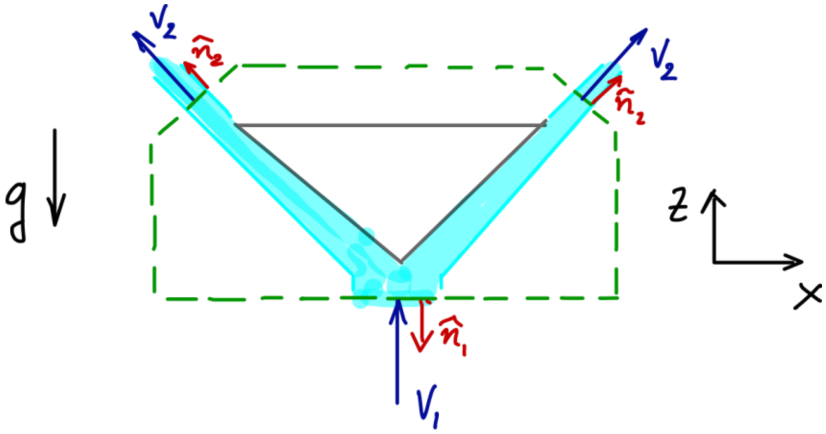
$$\Longrightarrow \quad V_1 = V_2$$

Mass flux:

$$\dot{m} = \rho V_0 A_0 = \rho V_1 A_1 = \rho V_2 A_2$$

$$= 999 \, \text{kg}/\text{m}^3 \cdot 15 \, \text{m}/\text{s} \times \pi \, 0.0225^2 \, \text{m}^2$$

$$= \underline{\underline{23.8 \, \text{kg}/\text{s}}}$$



Momentum in z: $\int_{cs} w \rho \vec{V}\cdot\vec{n}\, dA = \Sigma F_z$

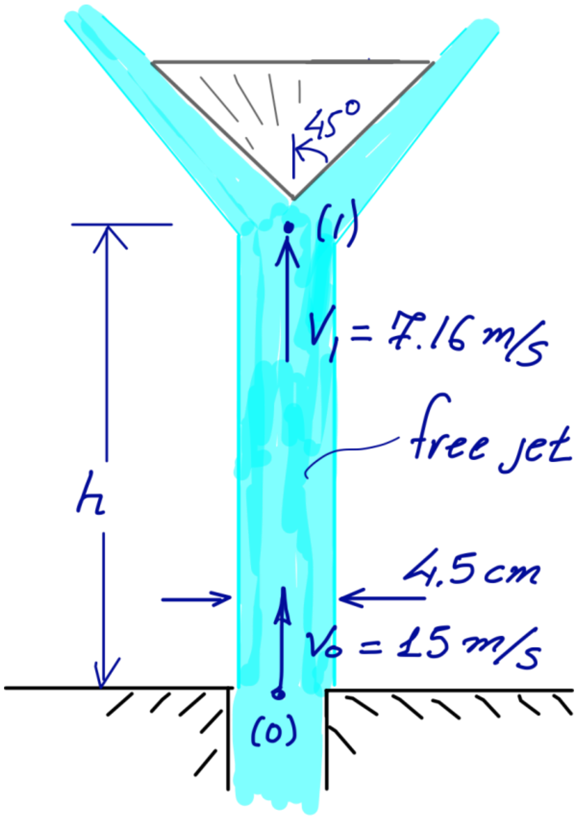
$$\Rightarrow \quad V_1 \rho (-V_1) A_1 + (V_2 \cos 45)\cdot \rho \cdot V_2 A_2 = -W_c - \underset{\to 0}{\cancel{W_w}}$$

$$\Rightarrow \quad -V_1 \underbrace{(\rho V_1 A_1)}_{\dot{m}} + V_2 \cos 45 \underbrace{(\rho V_2 A_2)}_{\dot{m}} = -W_c$$

$$\Rightarrow \quad \dot{m} \left(-V_1 + V_2 \cos 45\right) = -W_c \qquad (V_1 = V_2)$$

$$\Rightarrow \quad \dot{m} \left(1 - \cos 45\right) V_1 = W_c$$

$$\Rightarrow \quad V_1 = \frac{W_c}{\dot{m}\,(1-\cos 45)} = \frac{50\,N}{23.8\,kg/s \cdot (1-\cos 45)} = \underline{\underline{7.16\,m/s}}$$



Apply Bernoulli between (0) and (1)

$$\underset{\to 0}{\cancel{P_0}} + \frac{1}{2}\rho V_0^2 + \gamma z_0 = \underset{\to 0}{\cancel{P_1}} + \frac{1}{2}\rho V_1^2 + \gamma z_1$$

$$\Rightarrow \quad \frac{1}{2}\rho\left(V_0^2 - V_1^2\right) = \gamma(z_1 - z_0) = \gamma h = \rho g h$$

$$\Rightarrow \quad h = \frac{V_0^2 - V_1^2}{2g} = \frac{15^2 - 7.16^2}{2 \times 9.81}\ m$$

$$\Rightarrow \quad \underline{\underline{h = 8.85\ m}}$$